A New Restricted Full-Rank Single-Symbol Decodable Design for Four Transmit Antennas

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Abstract—Recently, a single-symbol decodable transmit strategy based on preprocessing at the transmitter has been introduced to decouple the quasi-orthogonal space-time block codes (QOSTBC) with reduced complexity at the receiver [9]. Unfortunately, it does not achieve full diversity, thus suffering from significant performance loss. To tackle this problem, we propose a single diversity scheme with four transmit antennas in this letter. The proposed code is based on a class of restricted full-rank single-symbol decodable design (RFSDSD) and has many similar characteristics as the coordinate interleaved orthogonal designs (CIODs), but with a lower peak-to-average ratio (PAR).

Index Terms—Coordinate interleaved orthogonal designs, full diversity, quasi-orthogonal space-time block code, restricted full-rank single-symbol decodable design.

I. INTRODUCTION

THE space-time block codes (STBCs) obtained from orthogonal designs (ODs) [1], [2] provide a promising transmission scheme in multiantenna systems due to their full diversity and single-symbol decodability (symbol-by-symbol decoding). However, it is proved in [2] that their symbol rates are upper bounded by 3/4 when complex signal constellations and more than two transmit antennas are used. To increase the rate while preserving the full diversity, two classes of single-symbol decodable (SSD) STBCs have been proposed: 1) coordinate interleaved orthogonal designs (CIODs) [3] and 2) minimum-decoding-complexity (MDC) STBCs from quasi-ODs (QODs) [4], [5]. Recently, as an extension to ODs and CIODs that allow single-symbol decoding, Sundar Rajan et al. proposed a so-called unrestricted full-rank single-symbol decodable design (UFSDD) and restricted full-rank single-symbol decodable design (RFSDSD), respectively, in [6]. However, to the best of our knowledge, the CIODs or generalized CIODs (GCIODs) [6] are the only codes that were found satisfying restricted FSDD (RFSDS). In this letter, we obtain a new class of RFSDSD other than CIODs or GCIODs.

Let us consider a $1 \times N_t$ (we assume $T = N_t$) linear STBC given by

$$C = \sum_{k=0}^{K-1} s_{kI} A_{2k} + s_{kQ} A_{2k+1}$$

(1)

where $s_k = s_{kI} + j s_{kQ}, k = 0, \ldots, K - 1$, are the $K$ complex variables with $s_{kI}$ and $s_{kQ}$ denoting, respectively, the real and imaginary part of $s_k$, and $j = \sqrt{-1}$. $N_t$ denotes the number of transmit antennas, and $T$ is the number of time slots for one codeword. $\{A_k\}_{k=0}^{2K-1}$ is a set of $T \times N_t$ complex matrices called weight (dispersion) matrices of $C$.

The following important conditions were introduced in [6] to classify single-symbol decodable STBCs:

$$A_k^H A_k = 0, \quad 0 \leq k \leq K - 1$$

(2)

$$A_k^H A_{k+1} = 0, \quad 0 \leq k \leq K - 1$$

(3)

where $(\cdot)^H$ stands for the complex conjugate transpose of matrix $(\cdot)$, and expression (4) specifies $A_k^H A_k$ as a full-rank matrix for all $k$. A linear STBC is UFSDD if and only if all three conditions (2)–(4) are satisfied. On the other hand, a linear STBC is RFSDSD if and only if the following three conditions are satisfied:

1) the weight matrices satisfy conditions (2) and (3), but not satisfying (4);
2) $A_k^H A_{2k} + A_k^H A_{2k+1}$ is full-rank for all $0 \leq k \leq K - 1$;
3) the coordinate product distance (CPD) [6] of the signal set $A$ is nonzero.

The single-symbol decodable STBCs presented in [4], [5], [7], and [8] satisfy (2) and (4), but not (3). Thus, they are a new class of codes which differ from UFSDD and RFSDSD.

By utilizing the fact that the eigenvectors of the equivalent channel are fixed and independent from the channel realizations, a single-symbol decodable transmit strategy based on preprocessing has been proposed in [9] and [10] for the quasi-orthogonal space-time block codes (QOSTBC). For convenience, we call this scheme SSD [9]. However, the performance loss of SSD [9] is significant as it does not achieve the full diversity order.

In this letter, we derive a generic algebraic structure for systems with $N_t = 4$ in order to gain a further insight into SSD [9]. From the structure, we discover that the scheme SSD [9] actually satisfies condition 1). In light of this finding, we propose a full diversity design scheme for SSD [9] in accordance with the conditions 2) and 3); thus, the performance can be improved greatly in comparison to SSD [9]. It is a class of new RFSDSD code as the all three conditions of RFSDSD are satisfied. The
proposed code shares many characteristics with CIOD but has a lower peak-to-average ratio (PAR), which is the major advantage of the proposed code over CIOD.

II. System Model

We consider a MIMO system with $N_t = T = 4$ and $N_r$ receive antennas. The system model is given by

$$\mathbf{Y} = \mathbf{C} \mathbf{H} + \mathbf{N}$$

(5)

where $\mathbf{Y} = [y_{tq}]_{T \times N_r}$ is the received signal matrix whose entry $y_{tq}$ is the signal received at antenna $q$ at time $t$, where $t = 1, 2, \ldots, T$, and $q = 1, 2, \ldots, N_r; \mathbf{N} = [n_{tq}]_{T \times N_r}$ is the noise matrix; $\mathbf{C} = [c_{tp}]_{T \times N_t}$ is the transmitted signal matrix whose entry $c_{tp}$ is the signal transmitted at antenna $p$ and at time $t$, where $t = 1, 2, \ldots, T$, and $p = 1, 2, \ldots, N_t$. $\mathbf{H} = [h_{tpq}]_{N_t \times N_r}$ is the channel matrix whose entry $h_{tpq}$ is the channel coefficient from transmit antenna $p$ to receive antenna $q$. The entries of the matrices $\mathbf{H}$ and $\mathbf{N}$ are mutually independent, zero-mean, and complex Gaussian random variables. $h_{tpq}$ has unit variance and $n_{tpq}$ has variance $N_0/\rho$, where $\rho$ is the signal-to-noise ratio (SNR) per receive antenna. The channel is assumed to be flat fading and remains constant for a block of $T$ symbols and changes independently from block to block. It is further assumed that the transmitter has no channel state information (CSI) and the receiver has perfect CSI.

In the following, we focus on the $4 \times 4$ quasi-orthogonal space-time block codes (QOSTBCs) [11] and briefly review the single-symbol decodable transmit strategy proposed in [9]. The codeword is given by

$$\mathbf{C} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2^* & x_1^* & -x_3 & -x_4^* \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4^* & -x_3^* & x_2 & -x_1 \\ \end{bmatrix}$$

(6)

which is a function of the vector $\mathbf{x} = [x_1 \; x_2 \; x_3 \; x_4]^T$. After rearranging and complex-conjugating some rows of $\mathbf{Y}$, (5) can be reformulated as

$$\mathbf{y}' = \mathbf{H}' \mathbf{x} + \mathbf{n}'$$

(7)

where $\mathbf{y}' = \begin{bmatrix} (\mathbf{y}_1)^T, (\mathbf{y}_2)^T, \ldots, (\mathbf{y}_{N_r})^T \end{bmatrix}^T$, $\mathbf{H}' = \begin{bmatrix} (\mathbf{H}_1)^T, (\mathbf{H}_2)^T, \ldots, (\mathbf{H}_{N_r})^T \end{bmatrix}^T$, and $\mathbf{n}' = \begin{bmatrix} (\mathbf{n}_1)^T, (\mathbf{n}_2)^T, \ldots, (\mathbf{n}_{N_r})^T \end{bmatrix}^T$. $\mathbf{n}_q = [n_{1q} \; n_{2q}^* \; n_{3q} \; n_{4q}^*]^T$, $q = 1, 2, \ldots, N_r$. $(\mathbf{\cdot})^T$ denotes the transpose of matrix $(\mathbf{\cdot})$, and $(\mathbf{\cdot})^*$ denotes the conjugate of a complex scalar. $\mathbf{y}_q; \mathbf{H}_q$ are given, respectively, as

$$\mathbf{y}_q = [y_{1q} \; y_{2q}^* \; y_{3q} \; y_{4q}^*]^T$$

$$\mathbf{H}_q = \begin{bmatrix} h_{1q} & h_{2q} & h_{3q} & h_{4q} \\ -h_{2q}^* & h_{1q}^* & -h_{3q} & h_{4q} \\ -h_{3q}^* & h_{4q} & h_{1q} & -h_{2q} \\ h_{4q}^* & -h_{3q}^* & -h_{2q}^* & h_{1q} \\ \end{bmatrix}.$$  

We get the equivalent channel matrix $\mathbf{H}' \mathbf{H}'$ as

$$\mathbf{H}' \mathbf{H}' = \begin{bmatrix} \alpha_1 & 0 & 0 & -\alpha_2^j \\ 0 & \alpha_1 & 0 & -\alpha_2 j \\ -\alpha_2 j & 0 & \alpha_1 & 0 \\ 0 & \alpha_2 j & 0 & \alpha_1 \end{bmatrix}$$

(8)

where $\alpha_1$ and $\alpha_2$ are defined as

$$\alpha_1 = \sum_{q=1}^{N_r} \sum_{p=1}^{4} |h_{pjq}|^2$$

$$\alpha_2 = \sum_{q=1}^{N_r} 2 \text{Re}(h_{1q}^* h_{3q} + h_{4q}^* h_{2q}).$$

(9)

We use the $\text{Re}(\mathbf{\cdot})$ and $\text{Im}(\mathbf{\cdot})$ to denote the real and imaginary parts of a complex scalar, respectively.

The symmetric matrix in (8) has the following singular value decomposition (SVD) as

$$\mathbf{H}' \mathbf{H}' = \mathbf{V} \mathbf{D} \mathbf{V}^H$$

(10)

where

$$\mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -j \\ 0 & 1 & -j & 0 \end{bmatrix}$$

and

$$\mathbf{D} = \text{diag} \left( \begin{bmatrix} \sqrt{\alpha_1 + \alpha_2} & \sqrt{\alpha_1 - \alpha_2} \\ \sqrt{\alpha_1 - \alpha_2} & \sqrt{\alpha_1 + \alpha_2} \end{bmatrix} \right).$$

An important characteristic is that the matrix $\mathbf{V}$ is a constant unitary matrix for arbitrary channel realizations. If we choose $\mathbf{V}$ as a preprocessing matrix in the transmitter, the vector $\mathbf{x}$ in (7) can be rewritten as

$$\mathbf{x} = \mathbf{V} \mathbf{s}$$

(11)

where $\mathbf{s} = [s_1 \; s_2 \; s_3 \; s_4]^T$, and $s_1, s_2, s_3, s_4 \in \mathbb{A}, \mathbb{A}$ is a complex signal set with unit average power. Combining expressions (7), (10), and (11) results in a completely decoupled model as follows:

$$\mathbf{y}'' = \mathbf{D} \mathbf{s} + \mathbf{w} = \mathbf{D}^{-1} \mathbf{V} \mathbf{H}' \mathbf{H}' \mathbf{y}'$$

(12)

where $\mathbf{w}$ is the noise vector, whose entries are mutually i.i.d. complex Gaussian random variables with zero-mean and identical variance.

III. Proposed RFSDU Code

A single-symbol decodable transmit strategy SSD has been derived in [9] based on the completely decoupled model expressed by (12). Unfortunately, SSD [9] does not achieve the full diversity provided by the MIMO channel, which will be proven below.

A. Analysis of SSD [9]

Substituting (11) into (6) and combining with (1), we get the weight matrices of SSD [9] with $K = 4$ as follows:

$$\mathbf{A}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -j & 0 \\ 0 & -1 & 0 & -j \\ -j & 0 & -1 & 0 \\ 0 & j & 0 & -1 \end{bmatrix}$$

$$\mathbf{A}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} j & 0 & 1 & 0 \\ 0 & j & 0 & -1 \\ 1 & 0 & -j & 0 \\ 0 & 1 & 0 & j \end{bmatrix}$$
\( A_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & j \\ 1 & 0 & -j & 0 \\ 0 & -j & 0 & 1 \\ -j & 0 & -1 & 0 \end{bmatrix} \)

\( A_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & j & 0 & -1 \\ -j & 0 & -1 & 0 \\ 0 & 1 & 0 & j \\ -1 & 0 & j & 0 \end{bmatrix} \)

\( A_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & j & 0 \\ 0 & -1 & 0 & j \\ j & 0 & -1 & 0 \\ 0 & -j & 0 & -1 \end{bmatrix} \)

\( A_5 = \frac{1}{\sqrt{2}} \begin{bmatrix} j & 0 & -1 & 0 \\ 0 & j & 0 & 1 \\ -1 & 0 & -j & 0 \\ 0 & -1 & 0 & j \end{bmatrix} \)

\( A_6 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & -j \\ 1 & 0 & j & 0 \\ 0 & j & 0 & 1 \\ j & 0 & -1 & 0 \end{bmatrix} \)

\( A_7 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & j & 0 & 1 \\ -j & 0 & 1 & 0 \\ 0 & -1 & 0 & j \\ 1 & 0 & j & 0 \end{bmatrix} \) \tag{13}

It is easy to verify that all the weight matrices satisfy condition 1) but not conditions 2) and 3), so that only partial diversity can be obtained no matter how we design CPD. Next, we present a regrouping scheme to make the code satisfying condition 2).

### B. Regrouping Scheme to Satisfy Condition 2)

We group all the weight matrices \( {\mathbf{A}}_{k} = \{ A_{p}^{H} {A}_{q} \} \), \( k = 0, 1, 2, 3 \) and \( p, q = 0, 1, \ldots, 7, p \neq q \). The \( A_{p} \) and \( A_{q} \) in \( G_{k} \) are the coefficients matrices of \( s_{k1} + j s_{kQ} \in \mathbb{A} \), \( k = 0, 1, 2, 3 \). For example, the expression (1) shows that the groups are \( g_{k} = \{ A_{k0}, A_{k1} \}, g_{0} = \{ A_{0}, A_{1} \}, g_{1} = \{ A_{2}, A_{3} \}, g_{2} = \{ A_{4}, A_{5} \}, g_{3} = \{ A_{6}, A_{7} \} \).

Another interpretation of condition 2) is that for any group \( g_{k} = \{ A_{p}^{H} {A}_{q} \}, k = 0, 1, 2, 3 \) and \( p, q = 0, 1, \ldots, 7, p \neq q \), it is necessary for the matrix \( A_{p}^{H} {A}_{p} + A_{q}^{H} {A}_{q} \) to have full-rank. Next, we will regroup the weight matrices to derive a new code that satisfies condition 2).

After examining all possible groups for weight matrices, we obtain

\[
\text{Rank} \left( A_{p}^{H} {A}_{p} + A_{q}^{H} {A}_{q} \right) = \begin{cases} 2, & p, q \in N_{1} \text{ or } p, q \in N_{2} \\ 4, & p \in N_{1}, q \in N_{2} \text{ or } p \in N_{2}, q \in N_{1} \end{cases} \tag{14}
\]

where \( N_{1} = \{ 0, 1, 2, 3 \} \) and \( N_{2} = \{ 4, 5, 6, 7 \} \). In Table I, we denote \( \Delta \) as the two weight matrices that satisfy (15).

It is noteworthy that there are many grouping patterns to satisfy condition 2). As an example, we choose \( g_{0} = \{ A_{0}, A_{1} \}, g_{1} = \{ A_{2}, A_{3} \}, g_{2} = \{ A_{4}, A_{5} \}, g_{3} = \{ A_{6}, A_{7} \} \) shown as “\( \Delta \)” in Table I. The new code is obtained as

\[
C' = s_{0}A_{7} + s_{0}Q_{A} + s_{1}A_{5} + s_{1}Q_{A} + s_{2}A_{2} + s_{2}Q_{A_{4}} + s_{3}A_{1} + s_{3}Q_{A_{6}}. \tag{16}
\]

It is easy to prove that the group has no effect on condition 1). Combining (13) and (16), the code can be rewritten as (17) at the bottom of the page, which is an interleaved version of SSD [9] and satisfies both conditions 1) and 2).

### C. Optimal Constellation Design

Condition 3) can always be satisfied by rotating a given signal set \( A \). In order to design an optimal rotation angle to maximize the coding gain, we first derive the determinant expression for the codeword distance matrix, from which the coding gain can be completely determined. Based on the codeword definition in (16), we let

\[
C' = C'(s_{0}, s_{1}, s_{2}, s_{3}) \text{ and } C'' = C'(s_{0}', s_{1}', s_{2}', s_{3}')
\]

\[
\Delta C = C' - C' \neq 0.
\]

Combining (13) and (16), we obtain

\[
\det(\Delta C^{H} \Delta C) = \left( 4 \Delta s_{0}^{2} + \Delta s_{1}^{2} + \Delta s_{2}^{2} + \Delta s_{3}^{2} \right) \tag{18}
\]

\[
\cdot \left( \Delta s_{0}^{2} + \Delta s_{1}^{2} + \Delta s_{2}^{2} + \Delta s_{3}^{2} \right)
\]

where \( \Delta s_{k} = s_{k} - \hat{s}_{k} \), \( \Delta s_{k}^{2} = s_{k}^{2} - \hat{s}_{k}, \) \( \hat{s}_{k} = s_{k} + j s_{kQ} \), \( \hat{s}_{k} = s_{k} - j s_{kQ} \), and at least one \( s_{k} \) differs from \( \hat{s}_{k}, k = 0, 1, 2, 3 \).

To obtain the minimum of determinant expression in (18), without loss of generality, we assume \( s_{k} \neq \hat{s}_{k} \) only for \( k = 0 \). Then (18) can be rewritten as

\[
\det(\Delta C^{H} \Delta C) = \left( 4 \Delta s_{0}^{2} \Delta s_{0}^{2} \right). \tag{19}
\]
Consequently, maximization of the coding gain is equivalent to maximization of the CPD of signal set $\mathcal{A}$, which is similar to CIODs. The optimal angle of rotation for CIODs with all different constellations is directly applicable to the code in (17). As an example, the optimal angle of rotation for square QAM is $\theta_{opt} = \arctan(2)/2 = 31.7175^\circ$.

Considering the CIOD shown in [6, (85)], we can derive its determinant expression for codeword distance matrix as
\[
\det(\Delta C^H \Delta C) = ((\Delta s_{Q1}^2 + \Delta s_{Q1}^2 + \Delta s_{Q1}^2 + \Delta s_{Q1}^2) \\
\cdot (\Delta s_{Q1}^2 + \Delta s_{Q1}^2 + \Delta s_{Q1}^2 + \Delta s_{Q1}^2))^2
\]
which differs from (18) only in one coefficient. However, the energy in (20) is double of that in (18) because only half of transmit antennas are utilized for CIOD. Therefore, the proposed code has the same coding gain as CIOD.

D. ML Decoding

It can be shown that the ML decoding metric can be calculated as the sum $f_0(s_0) + f_1(s_1) + f_2(s_2) + f_3(s_3)$, where
\[
\begin{align*}
    f_0(s_0) &= (y_Q^y - d_{s_01})^2 + (y_{d_1}^y - d_{s_01})^2 \\
    f_1(s_1) &= (y_Q^y - d_{s_11})^2 + (y_{d_1}^y - d_{s_11})^2 \\
    f_2(s_2) &= (y_Q^y - d_{s_21})^2 + (y_{d_1}^y - d_{s_21})^2 \\
    f_3(s_3) &= (y_Q^y - d_{s_31})^2 + (y_{d_1}^y - d_{s_31})^2
\end{align*}
\] (21)
where $d_1 = \sqrt{\alpha_1 + \alpha_2}$, $d_2 = \sqrt{\alpha_1 - \alpha_2}$, and we see the equation at the bottom of the page.

IV. NUMERICAL RESULTS

The simulation results are presented in this section for different codes with four transmit antennas. All the codes employ QPSK constellation. The rotation angle is set to $31.7175^\circ$ for both CIOD and the proposed code. The bit-error-rate (BER) performance for SSD [9], CIOD, and the proposed code are shown in Fig. 1. It is observed that the proposed design improves the performance of the original SSD [9] significantly. About 5 dB of performance gain is obtained at BER = $10^{-4}$ with one receive antenna. The proposed RFSDD code has identical performance to CIOD. Similar observations hold for the case with two receive antennas.

V. CONCLUSION

The proposed code in this letter is a full diversity version of SSD [9]. It is a new RFSDD, and its coding gain and optimal angle of constellation rotation are the same as CIOD. Simulation results show that the proposed code has a more rapid BER slope than SSD [9] and achieves the same performance as CIOD. However, the proposed code has more dispersive power and lower peak-to-average ratio (PAR) than CIOD since half of the antennas are idle [6] in the latter case.

REFERENCES


Fig. 1. Simulation results of SSD [9], CIOD, and proposed code for four transmit antennas.

\[
\begin{bmatrix}
\text{Re}(y_i^y) \\
\text{Im}(y_i^y)
\end{bmatrix} = \begin{bmatrix}
y_1^y \\
y_2^y \\
y_3^y \\
y_4^y \\
y_5^y \\
y_6^y \\
y_7^y \\
y_8^y
\end{bmatrix}
\]